FERMAT'S LAST THEOREM

SYNOPSIS

This is an improved version of an earlier treatise. It features a short-form proof that $x^n + y^n = z^n$ is impossible in integers for $n > 2$. It is the writers belief that this is Fermat's original proof. To begin, a model for squared numbers will be introduced and used to devise a method to create all Pythagorean ($x^2 + y^2 = z^2$) relationships. Three equations will be derived from this process which indicate the existence of a Pythagorean equation in the model for squared numbers.

A model for higher powers of n will then be introduced. This model will be an extension of the model for squared numbers. Simple manipulations of this model will show that the end game packaging of quantities postulated to be x^n and y^n into spaces known to be x^n and y^n requires that *x*, *y*, and *z* form a Pythagorean equation! This is totally incompatible with the postulation that $x^n + y^n = z^n$ where $n > 2$. The proof is thus Reductio ad Absurdum.. The recognition of the before-mentioned three equations in the packaging process is the essence of the proof.

A number of hypothetical objections and rebuttals are included at the end of the appendix.

FORWARD

Validating Fermat's assertion that he had a proof is of utmost importance. It will never be known for sure that he had a proof in his own day and age until some credible effort can be put together to substantiate this. Also, it is possible that a short-form proof may provide valuable adjuncts to the recent accomplishments of "Twentieth Century Mathematics". The science of mathematics will have an intolerable void until Fermat's proof is discovered or it is proved that a short form proof cannot exist. This treatise will provide that proof.

Fermat's original proof is still the Holy Grail of mathematics.

We have no direct knowledge of what might have been Fermat's general approach; however, I think we have a few tantalizing clues from the notes he is said to have made in the margin of a text.

1. He was studying Diophantine analysis.

2. He conjectured about the possibility that the sum of two cubes could equal a third cube (in integers, of course).

3. He rapidly asserted that this was impossible.

4. With seemingly equal rapidity he dismissed the possibility of there being a solution in integers for $x^n + y^n = z^n$ for any $n > 2$!

5. He claimed to have a marvelous demonstration of this that was "Too long to include in the margin". It is as if he composed the entire proof in one sitting!!

All of this bespeaks for a proof with the following characteristics:

1. It must be quite brief, perhaps only a few pages.

2. It was inspired by the Diophantine literature he was studying.

3. It required the recognition of a commonality in all $n > 2$ equations. How else could he have proceeded so rapidly to his final statement?

4. It probably has several plateaus of logic which are simple in themselves, but obtuse in their application.

This treatise contains four sections which reconstruct what might have been Fermat's methodology in proceeding to a statement of his last theorem.

1. Establishment of a model for squared numbers.

2. The use of this model to generate all Pythagorean ($n = 2$) relationships and to identify a triad of equations which indicate the existence of a Pythagorean in x, y, and z when the equations occur in the model for squared numbers.

3. The creation of an advanced model for $n > 2$ powers of *x*, *y*, and *z*.

4. The use of this model to show that packaging postulated values of higher ($n > 2$) powers of *x* and *y* into spaces known to be actual values results in a Pythagorean ($n = 2$) in *x*, *y*, and *z*; which is absurd and thereby proves the theorem.

MODEL FOR SQUARED NUMBERS

							2	
						2	2	
					2	2	2	
				2	2	2	2	
			2	2	2	2	2	
		2	2	2	2	2	2	
	2	2	2	2	2	2	2	
1	1	1	1	1	1	1	1	
1	2	3	4	5	6	7	8	⟵ Number to be squared
1	4	9	16	...	...	...	64	⟵ Cumulative value of model

Figure 1: Model for squared numbers

This model consists of a horizontal base of ones and an upper structure of twos, thereby forming a triangular shape. The vertical dimension of the triangle equals the horizontal dimension. The cumulative value of the contents of the triangle equals the square of the horizontal dimension. This model enables easy visualization of squares, and (as will be seen later) of higher powers.

Throughout this treatise, the model will be referred to as a triangle. Sometimes it will be called an *r*, *x*, *y*, or *z* triangle depending on the horizontal dimension. Wherever meaningful, it may also be called an r^2, x^2, y^2, or z^2 triangle.

CHARACTERISTICS OF PYTHAGOREAN RELATIONSHIPS

Consider a known Pythagorean, x = 8, y = 15, and z = 17 as shown in Figure 2.

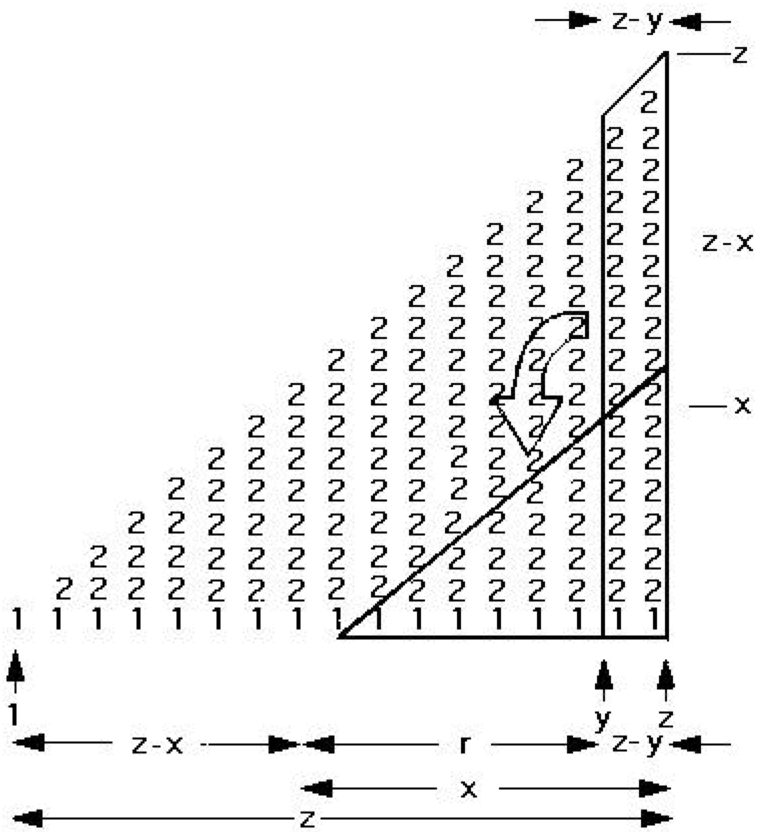

Figure 2: From Pythagorean equation, $8^2 + 15^2 = 17^2$.

The sum of the contents of the large overall triangle is z^2. The sum of the contents of the outlined trapezoid must equal x^2 since the remainder of the large triangle is y^2. Note the rhomboid atop the trapezoid with side dimensions, $(z - x)$ which equals 9, and $(z - y)$ which equals 2. This rhomboid encloses a group of twos. For the trapezoid to form an x^2 triangle (as in the model of Figure 1), the sum of the rhomboid contents must spill down and form the sum of the contents of the small triangle (with dimension *r*).

This small triangle and the lower portion of the trapezoid now equal x^2. Note that *r* equals 6 and r^2 equals 36 which is the sum of the rhomboid contents, $2(9)(2)r$ is even since r^2 is composed from the rhomboid contents of "twos". These generalized equations are obtained from the dimensions below Figure 2.

$$x = r + (z - y)$$
$$z = r + (z - x) + (z - y)$$

Also, $r^2 = 2(z - x)(z - y)$ since r^2 equals the sum of the rhomboid contents.

Figure 3 shows the process repeated with the trapezoid equal to y^2.

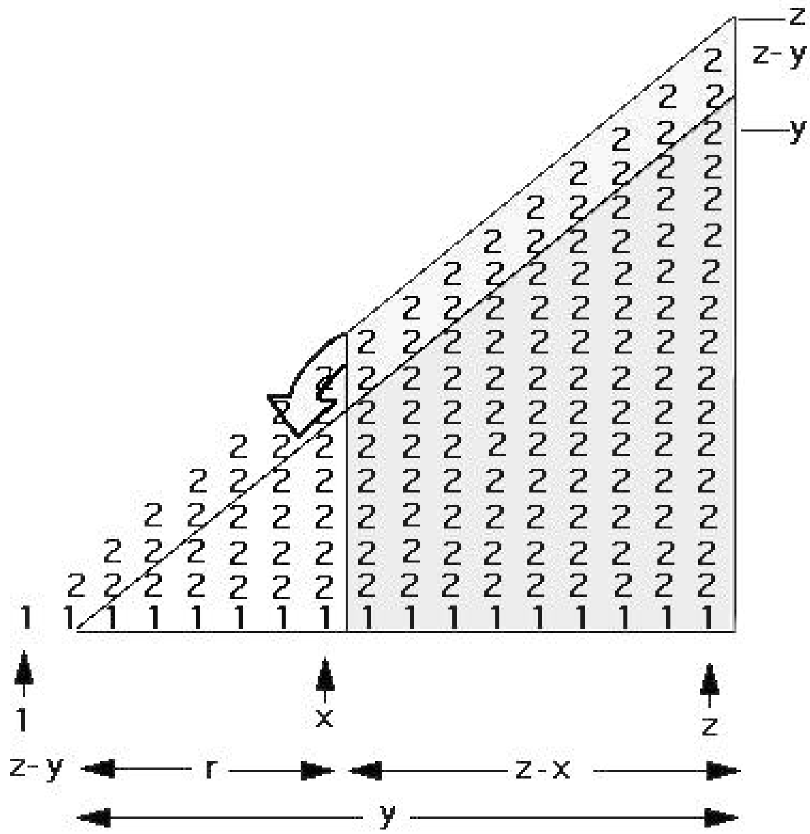

Figure 3: From Pythagorean equation, $8^2 + 15^2 = 17^2$.

Again, the sum of the rhomboid contents spills down and forms the sum of the contents of the small triangle with dimension, r, the same as before since the rhomboid dimensions are still $(z-x)$ and $(z-y)$. The sum of the contents of the *r* triangle is again $2(z-x)(z-y)$. The following generalized equations are observed below Figure 3.

$$y = r + (z-x)$$
$$z = r + (z-x) + (z-y)$$
$$r^2 = 2(z-x)(z-y)$$

As a result of the operations shown in Figures 2 and 3, the following very important equations are identified in generalized form:

(1) $x = r + (z - y)$

(2) $y = r + (z - x))$

(3) $z = r + (z - x) + (z - y)$

(4) $r^2 = 2(z - x)(z - y)$

It will now be shown that when (1), (2), and (3) occur in the model for squared numbers with overall dimension, z, a Pythagorean relationship in x, y, and z exists.

The operations shown in Figures 2 and 3 have shown how the contents of the rhomboids in a known Pythagorean relationship spill down to form triangles whose contents equal r^2.

Pythagorean relationships in x, y, and z can be created by a reverse process wherein an even number, r, is selected, squared, and factored to generate quantities which represent the sides of rhomboids. These can then used to calculate x, y, and z by using equations(1), (2), and (3) with actual numbers in place of the generalized values, r, (z-x), and (z-y). Henceforth, r will be called the root number since it is the starting point for creating Pythagoreans.

Appendix I shows this process in detail with numeric examples. Appendix I also clarifies how (1), (2), and (3) relate in general and numeric form.

Thus, in general terms, equations (1), (2), and (3) generate a Pythagorean from a root number, *r*, where $r^2 = 2(z - x)(z - y)$.

Is there a corollary to this triad of equations? That is, does the presence of equations (1), (2), and (3) within the model for squared numbers indicate the existence of a Pythagorean, $x^2 + y^2 = z^2$? The answer is yes by the following logic:

Suppose that ($z - x$) and ($z - y$) are factors of $\frac{r^2}{2}$ and are used in (1), (2), and (3) to generate *x*, *y*, and *z*. Then, *x*, *y*, and *z* are exactly as shown in the triad of equations. The corollary is actually obvious.

Corollary I: Equations (1), (2), and (3) are necessary and sufficient to prove the existence of a Pythagorean, $x^2 + y^2 = z^2$, when they appear in the triangular model for squared numbers.

Corollary I includes, implicitly, the requirement that *r* be even. Even so, a small proof for this is included at the end of Appendix I.

Equations (1), (2), and (3) identify one and only one Pythagorean which fits exactly into a given triangular model for squared numbers having the dimensions *x*, *y*, *z*, and *r*. It is important to realize that numbers for *r*, *x*, *y*, and *z* cannot simply be selected and inserted in equations (1), (2), and (3) to generate Pythagoreans. The results would not fit into the model and could only generate a Pythagorean by happenstance. The process wherein *r* is squared and factored to obtain $(z-x)$ and $(z-y)$ is implicit in the use of these equations.

MODEL FOR HIGHER POWERS

Consider the model for squares in Figure 1. A model for z^3 is easily formed by stacking z^3 triangles horizontally. Thus z^3 is a wedge-shaped figure whose width, height, and length are all equal to *z*. For higher powers of *z*, the length is z^{n-2} while the width and height remain equal to *z*.

Figure 4 shows this model. For simplicity, the twos and ones are not shown in the outlined forms. Also shown is how x^3 and y^3 can be represented if $x^3+y^3+z^3$ is postulated.

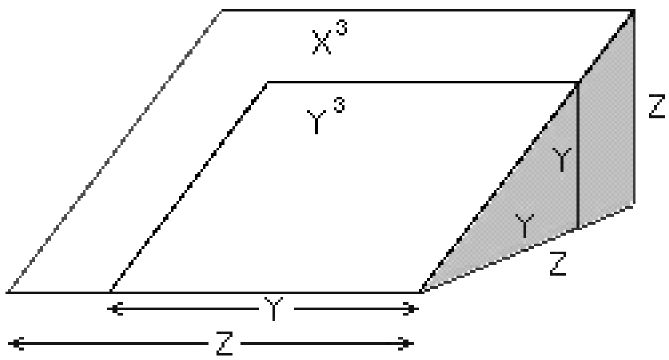

Figure 4: Model for higher powers of n. (Cubed values shown)

If $x^3+y^3+z^3$ then x^3 must be equal to the L-shaped form shown in Figure 4. Figure 5 shows this form separately.

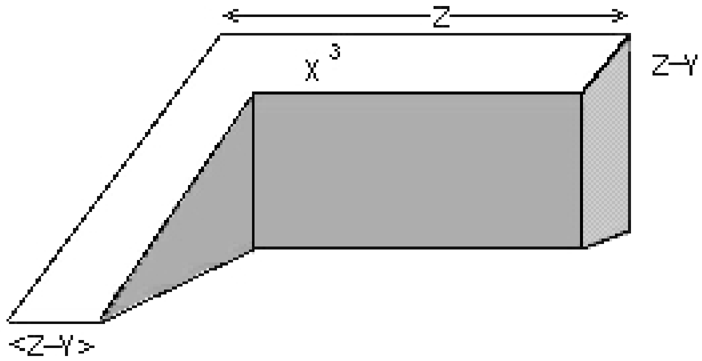

Figure 5: x^3 (postulated) form separated from the model.

Consider a separate z^3 wedge with the wedge representing x^3 removed, thereby leaving a void known to equal x^3. Now suppose that the form in Figure 5, postulated to be x^3, is placed snugly within this x^3 void as shown in Figure 6.

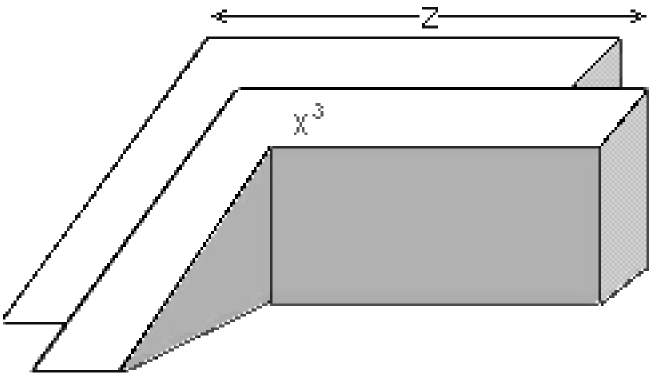

Figure 6. x^3 L-shaped form placed within x^3 void.

Note that parts of the L-shaped body (postulated to be x^3)protrude above, to the side, and to the front of the void representingx^3. Suppose that these protrusions are sliced away and the "detritus" is placed aside temporarily.

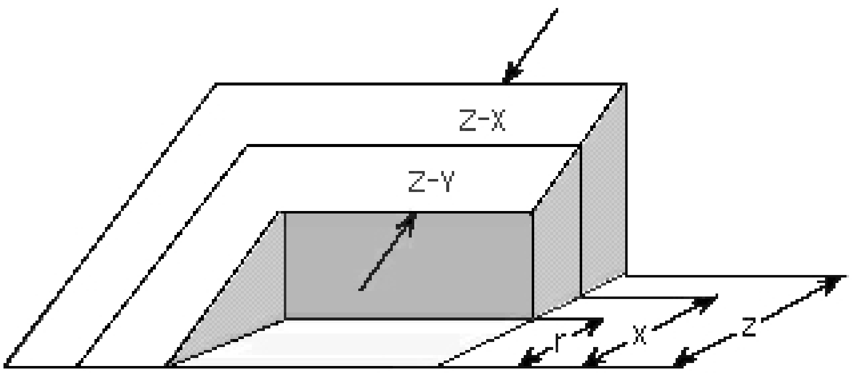

Figure 7: x^3 void partially filled in.

Figure 7 shows that the x^3 void has been partially filled by x^2 triangles and a trapezoidal form of thickness, $(z - y)$. The remaining space is a wedge-shaped void of r^2 triangles. Note that the foregoing process has retained the "ones" on the bottom of the advanced models, thereby maintaining the integrity of the model.

Now, in keeping with the postulate that $x^3 + y^3 + z^3$, the remaining detritus must exactly fit into the void of r^2 triangles to completely fill the x^3 void.

Since all of the r^2 triangles are exactly alike, we may confine the discussion to a single "z" triangle involving *r*, *z*, $(z - x)$, and $(z - y)$. When this is done the following equations become clear.

$$x = r + (z - y)$$
$$z = r + (z - x) + (z - y)$$

If the foregoing exercise is repeated with a y^3 L-shaped form filling a y^3 void, Figure 8 is obtained.

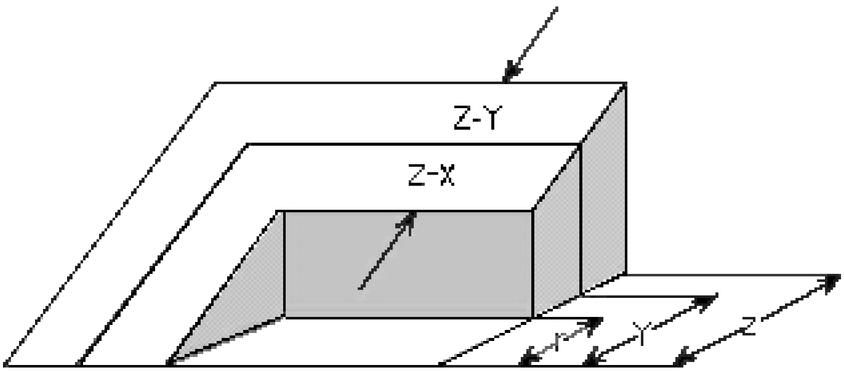

Figure 8. y^3 void partially filled in.

Again, in keeping with the postulate that $x^3 + y^3 + z^3$, the remaining detritus must exactly fit into the void of r^2 triangles to complete filling of the y^3 void. Now the following equations emerge:

$$y = r + (z - x)$$
$$z = r + (z - x) + (z - y)$$

The value of *r* is the same as that found in Figure 7 since $(z - x)$ and $(z - y)$ have simply changed positions in the model. Again, we may confine the discussion to a single "z" triangle since the *r*, *z*, $(z - x)$,and $(z - y)$ dimensions are exactly the same.

The two-stage filling of the x^3 and y^3 voids has yielded the information to complete the proof! The resultant equations are identical to the equations which define a Pythagorean in *x*, *y*, and *z* by corollary (1) namely:

(1) $x = r + (z - y)$
(2) $y = r + (z - x)$
(3) $z = r + (z - x) + (z - y)$

x, *y*, and *z* which were postulated to form a cubed relationship in the original model must now form a $n = 2$ relationship in the same model! This is clearly an absurdity. If $x^2 + y^2 + z^2$, then z^3 would greatly exceed $x^3 + y^3$ in the postulate. The cubed relationship is impossible in integers!!

The value of *n* in the postulation, $x^n + y^n + z^n$ is completely immaterial!! This is because the proof process has introduced a general characteristic of all of the higher powers of n, namely, values of $(z - x)$ and $(z - y)$ which, in the model, can relate to any value of n!!

Also, all of the detritus which would contain dimensions such as $(z^{n-2} - x^{n-2})$ has been buried with the placement of the L-shaped forms in their respective voids and filling the *r* triangles with the detritus. Thus, the proof process will always show that $x^2 + y^2 + z^2$ must exist no matter what value of *n* is postulated.

Therefore $x^n + y^n + z^n$ is impossible in integers for $n > 2$!!!

In a nutshell, this proof works because the proof process requires that *x*, *y*, and *z* in the relationship, $x^n + y^n + z^n$, must fit into a Pythagorean framework. This framework is a *z* triangle in the model for advanced powers ($n > 2$) of *x*, *y*, and *z*.

Appendix I: Finding all Pythagoreans

The following process is used to generate Pythagoreans in *x*, *y* and *z*.

1. Define an even root number in prime factors.
   Root number $= r = 2(p_1 p_2 p_3 p_4 \ldots)$

2. Square the root number to get the root square.
   Root square $= r^2 = 4(p_1 p_2 p_3 p_4 \ldots)^2$

   The root square must equal the sum of the contents of a rhomboid as shown in Figures 2 and 3 to create a Pythagorean.

3. Obtain the value of the rhomboid area (see Figures 2 and 3). Since the rhomboids are composed of twos, the root square must be divided by two to obtain the rhomboid area:

   Rhomboid area = $2(p_1p_2p_3p_4\ldots)^2$

4. Obtain the rhomboid sides by separating the rhomboid area into 2 factors. These will be the height and breadth of the rhomboid. One side will be even since it contains 2.

   Even factor $= 2(p_1 p_2 \ldots)^2$

   Odd factor $= (p_3 p_4 \ldots)^2$

   The primes may be grouped in any combination. Only one is shown above. The even and odd factors are the rhomboid sides,(z-x) and (z-y)

5. Generate *x*, *y*, and *z* from (1), (2), and (3) using actual numbers:

(1) $x = 2(p_1 p_2 p_3 p_4 \ldots) + (p_3 p_4)^2$

(2) $y = 2(p_1 p_2 p_3 p_4 \ldots) + 2(p_1 p_2 \ldots)^2$

(3) $z = 2(p_1 p_2 p_3 p_4 \ldots) + (p_3 p_4 \ldots)^2 + 2(p_1 p_2 \ldots)^2$

The most primitive root number, $2(1\times1)$, yields only one solution:

r^2	$\frac{r^2}{2}$	Even factor	Odd factor
$4(1\times1)^2$	$2(1\times1)^2$	$2(1)^2$	$(1)^2$

$x = 2(1\times1) + (1)^2 = 3$

$y = 2(1\times1) + 2(1)^2 = 4$

$z = 2(1\times1) + 2(1)^2 + (1)^2 = 5$

The root number, $2(2\times1)$ has only one valid solution

r^2	$\frac{r^2}{2}$	Even factor	Odd factor
$4(2\times1)^2$	$2(2\times1)^2$	$2(2)^2$	$2(1)^2$

$x = 2(2\times1) + (1) = 5$

$y = 2(2\times1) + 2(2)^2 = 12$

$z = 2(2\times1) + 2(2)^2 + (1)^2 = 13$

The prime, 2, in the even factor cannot be transposed to the odd factor because *x*, *y*, and *z* would then have a common factor, 2. The result would be a 6, 8, 10 Pythagorean which is a trivial doubling of the 3, 4, 5 solution.

The root number, $2(3\times 1)$ has two solutions since the prime, 3, may be transposed from the even to the odd factor.

r^2	$\frac{r^2}{2}$	Even factor	Odd factor
$4(3\times 1)^2$	$2(3\times 1)^2$	$2(3)^2$	$2(1)^2$

$$x = 2(3\times 1) + (1) = 7$$
$$y = 2(3\times 1) + 2(3)^2 = 24$$
$$z = 2(3\times 1) + 2(3)^2 + (1)^2 = 25$$

Transposing the factor 3, the following valid solution results:

$$x = 2(3\times 1) + (3) = 15$$
$$y = 2(3\times 1) + 2(1)^2 = 8$$
$$z = 2(3\times 1) + 2(2)^2 + (1)^2 = 13$$

The above examples are the smallest Pythagoreans. As the number of primes in the root number goes up, the number of factors increases rapidly. Consider $r = 2(1\times 3\times 5)$. The following Pythagoreans result:

Even Factors	Odd Factors	x	y	z
$2(1)^2$	$(3\times 5)^2$	255	32	257
$2(3)^2$	$(5)^2$	55	48	73
$2(5)^2$	$(3)^2$	39	80	89
$2(3\times 5)^2$	$(1)^2$	31	480	481

The root number, $2(3\times 5\times 7)$, yields eight Pythagoreans as follows:

Even Factors	Odd Factors	x	y	z
$2(1)^2$	$(3\times5\times7)^2$	11235	212	11237
$2(3)^2$	$(5\times7)^2$	1435	228	1453
$2(5)^2$	$(3\times7)^2$	651	260	701
$2(7)^2$	$(3\times5)^2$	435	308	533
$2(3\times5)^2$	$(7)^2$	259	660	709
$2(3\times7)^2$	$(5)^2$	235	1092	1117
$2(5\times7)^2$	$(3)^2$	219	2660	2669
$2(3\times5\times7)^2$	$(1)^2$	211	22260	22261

Note that the values of x, y, and z vary widely because the rhomboid changes shape with each different pair of factors. A long, slim rhomboid will result in large dimensions for *z* and either *x* or *y*. As the rhomboid dimensions become more equal, the *x* and *y* values become smaller and more equal, and the *z* value becomes smaller. The area of the rhomboid is always the same.

Proof that "*r*" is always an even number

The relationship (4), $x = r + (z - x) + (z - y)$, which occurs in both the model for squares and the model for $n > 2$, gives the following solution for r:

$$r = z - (z - x) - (z - y)$$
$$r = z - z + x - z + y$$
$$r = (x + y) - z$$

Assume that *x*, *y*, and *z* have no common factor. (since the original postulate, $x^n + y^n + z^n$ is in its lowest terms.)

1. If *z* is odd, then *x* and *y* cannot both be odd since the postulate would then be unequal. *x* and *y* must be "odd, even". Thus $r = (\text{odd} + \text{even}) - \text{odd} = \text{even}$

2. If z is even, then both x and y must be odd to form an even sum in the postulate. Thus $r = (\text{odd} + \text{odd}) - \text{even} = \text{even}$.

Thus r in $r = (x + y) - z$ will always be even for any $x^n + y^n + z^n$.

Objections and Rebuttals

Objection No.1: Equations (1), (2), and (3) don't define a Pythagorean. Numbers can be algebraically calculated and inserted into these equations which do not produce Pythagorean equations; therefore the proof is flawed.

Rebuttal: Equations (1), (2), and (3) must relate to the model for squared numbers which is used repeatedly throughout this treatise. Figures 2 and 3 and Appendix I show how these equations result from a root number, *r*, which is squared and factored to obtain both numerical and generalized quantities. The generalized quantities, $(z - x)$ and $(z - y)$, are used with "*r*" to generate Pythagorean relationships. It is not surprising that the efforts to plug numbers into the triad of equations do not produce a Pythagorean. There is one and only one_Pythagorean in *x*, *y*, and *z* which will fit into a given "z" triangle with root number, "*r*". This is the case where r^2 is equal to $2(z - x)(z - y)$. Appendix I shows how this quantity generates the triad of equations in both numerical and general form.

Objection No.2: The recognition of (1), (2), and(3) in the "end game" of the proof does not necessarily prove the existence of a Pythagorean in x, y, and z. The values of *x*, *y*, and *z* which are postulated to form relationships for $n > 2$ could be anything!! Why must they be represented by a Pythagorean equation which is generated from $r^2 = 2(z - x)(z - y)$?.

Rebuttal: Yes, they could be anything, but the proof process tells what they MUST be, and this overrides "could be". This is the essence of the "Reductioad Absurdum" process. Corollary I says that a (one and only) Pythagorean exists which is formed from a given root number, "*r*" and has the overall dimension, "z". Remember also, that if *r* squared is not equal to $2(z - x)(z - y)$, then the value of "*r*" in (1), (2), and (3) will be either larger or smaller, and the overall size of the "z" triangle will change! It will not fit into the postulated space in Figures 7, and 8!! The Pythagorean in *x*, *y*, and *z* is the only thing which will fit, but it is absurd!!

Objection No.3: How can this proof exist for all values of *n* greater than 2? How can this claim be made? How is this included in the proof process?

Rebuttal: The values of x, y, and z (and $(z - x)$ and $(z - y)$) are introduced into the proof process since they appear as dimensions at the sides of the advanced models and form the triad of equations which identify a Pythagorean. It doesn't matter what value of "n" is under consideration, the proof process is unchanged!!

Objection No.4: How do we know that this whole thing is not just an exercise in Pythagoreans? Is Fermat's proof really being addressed?

Rebuttal: The proof process brings in values of x, y, and z which are postulated to form relationships for $n > 2$. It doesn't matter what the value of n is; they all receive the same treatment. They must fit into the "frame work" of a Pythagorean relationship. This frame work is a "z" triangle described by equations (1), (2), and (3).

Objection No.5: Why must one accept Corollary I as true?

Rebuttal: Corollary I says in plain English that equations (1), (2), and (3) are the result, in general terms, of a root number which is squared, factored and used as shown in Appendix Ito yield this self-same triad of equations in the model for squared numbers. Therefore, If this same triad of equations occurs in the model for squared numbers (as it does in the "end game" of the proof),then a Pythagorean in x, y, and z exists. This is the absurdity which proves the theorem.